Vertex renormalization of weak interactions in compact stars: beyond leading order

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Neutrino emission rate from baryonic matter in neutron stars via weak neutral vector interaction is computed up to order $O(v_F^6)$, where v_F is the Fermi velocity in units of speed of light. The vector current polarization tensors are evaluated with full vertices which include resummed series in the particle-hole channel. The neutrino emissivity is enhanced compared to the $O(v_F^4)$ order up of 10% for values $v_F \leq 0.4$ characteristic to baryons in compact stars.

I. INTRODUCTION

The pair-breaking neutrino bremsstrahlung is a important process contributing to the neutrino cooling of a compact star. The rate of these processes was computed initially at one-loop [1–3] and, more recently, including vertex corrections [4–7]. Within the Standard Model the neutrino emission occurs via vector and axial-vector current interactions. The one-loop calculations suggest that the neutrino emission via neutral vector currents is large compared to the emission via axial vector currents. However, the vertex corrections substantially suppress the emission via vector currents [4–7] while they leave the axial vector emission unaffected. As a result, the neutrino emission via vector currents turns out to be subdominant compared to the same emission mediated by the axial-vector current interactions. Vertex corrections can be important in the context of neutrino scattering in proto-neutron-star matter, if the superfluidity sets in before the matter becomes transparent to neutrinos [8].

In this work we compute higher order corrections to the leading order (vanishing) contribution to the vector current neutrino emission rate. We adopt the same approach as in Ref. [5], where a non-zero temperature propagator formalism was used to compute the vertex corrections to the weak interaction rates in pair-correlated baryonic matter. The convenient techniques of Nambu-Gor'kov propagators were used for a re-summation of particle-hole ladder diagrams - an approach first developed by Abrikosov and Gor'kov in electrodynamics of superconductors [9] (see also Ref. [10]). In this theory the response of superconductors to external probes is expressed in the language of propagators at non-zero temperature and density with contact interactions that do not distinguish among the particle-hole and particle-particle channels. It is equivalent to the theories initially advanced by Bogolyubov, Anderson and others, which are based on the equations of motions for second-quantized operators. A more involved approach was developed subsequently on the basis of the ideas of Fermi-liquid theory for superconductors/superfluids by Refs. [11, 12]. The latter method implements wave-function-renormalization of quasiparticle spectrum and higher-order harmonics in the interaction channels, and postulates particle-hole (ph) and particle-particle (pp) interactions with different strength and/or sign. Application of this method in the context of neutrino emission and collective excitations can be found in Refs. [6, 13]. In the preceding paper [5] the driving terms in both channels were taken to be identical (i.e. $v_{pp} = v_{ph}$) and equal to the lowest order Landau parameter in baryonic matter. This identification for v_{ph} is consistent with the theory of normal Fermi-liquids and guarantees the correct limiting form of the response function in the unpaired state. The same identification for v_{pp} presumes pairing interaction which includes polarization of the medium in a manner that has been frequently used in the computations of neutron and neutron-star matter [14].

It is now well established that the pair-breaking process vanishes to the leading order (LO) in small momentum transfer for both neutral vector and axial-vector currents [1–7]. The purpose of this work is to revise and extend the results of Ref. [5] concerning the neutrino emission via neutral vector currents. First, we recompute and correct the next-to-leading (NLO) contribution. Second, we extend the calculation to the next-to-next-to-leading order (NNLO) contribution and access the convergence of the series.

The main focus and motivation of this work is the computation of the neutrino emissivity of baryonic matter in compact stars [15, 16]. The low-energy neutral weak current interaction Lagrangian describing the interaction of neutrino field ψ and baryonic current j_μ is given by

$$\mathcal{L}_W = -\frac{G_F}{2\sqrt{2}} j_\mu \bar{\psi}\gamma^\mu(1-\gamma^5)\psi, \tag{1}$$

where G_F is the Fermi coupling constant. The current of baryons for each B-baryon is

$$j_\mu = \bar{\psi}_B \gamma_\mu (c_V^{(B)} - c_A^{(B)}\gamma^5)\psi_B, \tag{2}$$

where ψ_B are the quantum fields of the baryons and $c_V^{(B)}$ and $c_A^{(B)}$ are the vector and axial vector couplings, respectively.

The rate at which neutrinos are radiated from matter (neutrino emissivity) is given by (for derivations using

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equilibrium or transport techniques see Refs. [2, 17])

$$\varepsilon_{\nu\bar{\nu}} = -2\left(\frac{G_F}{2\sqrt{2}}\right)^2 \int d^4q g(\omega)\omega \sum_{i=1,2}\int \frac{d^3q_i}{(2\pi)^3 2\omega_i} \times \mathrm{Im}[L^{\mu\lambda}(q_i)\,\Pi_{\mu\lambda}(q)]\delta^{(4)}(q-\sum_i q_i), \quad (3)$$

where $q_i = (\omega_i, \boldsymbol{q}_i)$, $i = 1, 2$ are the neutrino momenta, $g(\omega) = [\exp(\omega/T)-1]^{-1}$ is the Bose distribution function, $\Pi_{\mu\lambda}(q)$ is the retarded polarization tensor of baryons, and

$$L^{\mu\nu}(q_1, q_2) = 4\Big[q_1^\mu q_2^\nu + q_2^\mu q_1^\nu - (q_1\cdot q_2)g^{\mu\nu} - i\epsilon^{\alpha\beta\mu\nu}q_{1\alpha}q_{2\beta}\Big] \quad (4)$$

is the leptonic trace. Here the emissivity is defined per neutrino flavor, i.e., the full rate of neutrino radiation through weak neutral currents is larger by a factor N_f - the number of neutrino flavors. (We will assume $N_f = 3$ massless neutrino flavors.)

The paper is organized as follows. In Sec. II we set the stage and introduce the correlation functions needed for a description of the superfluid baryonic matter. In Sec. III we discuss the vertex functions and polarization tensors for neutral vector current interactions of baryons and neutrinos and their double expansion in small parameters of the theory. Section IV computes the neutrino emissivity via neutral vector currents up to NNLO order. Section V contains a brief discussion of the results. Some details of calculations are relegated to Appendix A and the differences to Ref. [5] are discussed in Appendix B.

II. PROPAGATORS

Consider low-density baryonic matter with attractive interaction in the ${}^1S_0$-channel. The interaction Lagrangian for the baryons is

$$\mathcal{L}_{int} = -v_{pp}\sum_{\boldsymbol{p}_1\neq\boldsymbol{p}_1'} \psi^\dagger_{B,\uparrow}(\boldsymbol{p}_1')\psi^\dagger_{B,\downarrow}(\boldsymbol{p}_2')\psi_{B,\downarrow}(\boldsymbol{p}_2)\psi_{B,\uparrow}(\boldsymbol{p}_1) \times \delta(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_1'-\boldsymbol{p}_2'), \quad (5)$$

where $v_{pp} < 0$ is the four-point coupling responsible for binding the particles in Cooper pairs. The imaginary-time momentum-space correlators are given by the 2×2 Nambu-Gor'kov matrix

$$\mathcal{G}_{\sigma,\sigma'}(i\omega_n,\boldsymbol{p}) = \begin{pmatrix} \hat{G}_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) & \hat{F}_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) \\ \hat{F}^+_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) & \hat{G}^+_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) \end{pmatrix}. \quad (6)$$

The elements of the matrix are time-order correlators of the baryon field ψ_B and $\psi_B^\dagger$; in the frequency-momentum domain these are given by [Ref. [5], Eqs. (5) and (6)]

$$\hat{G}_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) = \delta_{\sigma\sigma'}\left(\frac{u_p^2}{i\omega_n-\epsilon_p}+\frac{v_p^2}{i\omega_n+\epsilon_p}\right), \quad (7)$$

$$\hat{F}_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) = -i\sigma_y u_p v_p\left(\frac{1}{i\omega_n-\epsilon_p}-\frac{1}{i\omega_n+\epsilon_p}\right), \quad (8)$$

where $F^+_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) = F_{\sigma\sigma'}(i\omega_n,\boldsymbol{p})$, $\omega_n = (2n+1)\pi T$ is the fermionic Matsubara frequency, σ_y is the y component of the Pauli-matrix in spin space, $u_p^2 = (1/2)(1+\xi_p/\epsilon_p)$ and $v_p^2 = 1-u_p^2$ are the Bogolyubov amplitudes, ξ_p is the single particle energy in the unpaired state corresponding to the momentum p, and $\epsilon_p = \sqrt{\xi_p^2+\Delta_p^2}$ is the single particle energy in the paired state, with Δ_p being the (generally momentum- and energy-dependent) gap in the quasiparticle spectrum. The propagator for the holes is defined as $\hat{G}^+_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) = \hat{G}_{\sigma\sigma'}(-i\omega_n,-\boldsymbol{p})$. For an S-wave condensate the spin structure of the propagators can be made explicit by writing $\hat{G}_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) = \delta_{\sigma\sigma'}G(i\omega_n,\boldsymbol{p})$, $\hat{F}_{\sigma\sigma'}(i\omega_n,\boldsymbol{p}) = -i\sigma_y F(i\omega_n,\boldsymbol{p})$, *etc.*

The single-particle energies in the superconducting state, ϵ_p, are the solution of the Dyson-Schwinger equation $\mathcal{G}^{-1}_{\sigma,\sigma'} = \mathcal{G}^{-1}_{\sigma,\sigma',0} - \Sigma_{\sigma,\sigma'}$, where $\mathcal{G}_{\sigma,\sigma',0}$ is the diagonal free-particle propagator, whereas the self-energy $\Sigma_{\sigma,\sigma'}$ is a 2×2 matrix analogous to Eq. (6). The diagonal elements of the self-energy matrix renormalize the quasiparticle spectrum and the density of states on the Fermi surface in the normal state. We will assume that such a program has been carried out and write the particle energy in the unpaired state as $\xi_p = v_F(p-p_F)$, where v_F and p_F are the (effective) Fermi velocity and momentum. For contact pairing interactions (5) the gap function (the off-diagonal self-energy) is momentum-frequency independent; hereafter we set $\Delta_p \equiv \Delta$.

III. EFFECTIVE VERTICES AND POLARIZATION TENSORS FOR WEAK INTERACTIONS

A. Vertex functions

This section reviews and complements the discussion of the renormalization of the weak vertices in Ref. [5]. The diagrammatic approach and the mathematical structure of the theory is essentially the one found in Refs. [9–11], but its physical content (the electroweak dynamics of baryons) is distinct. We start by stating the equations that are obeyed by the weak vector-current vertex $\Gamma_\mu = (\Gamma_0, \vec{\Gamma})$ in pair-correlated matter. We use the convention in which greek indices run through temporal 0 and spatial 1,2,3 components, i.e., $\mu = 0, 1, 2, 3$. The latin indices run through spatial components only.

The temporal ($\mu = 0$) component of the vector-current vertex has four components in the Nambu-Gor'kov space in general. These components are given by the following

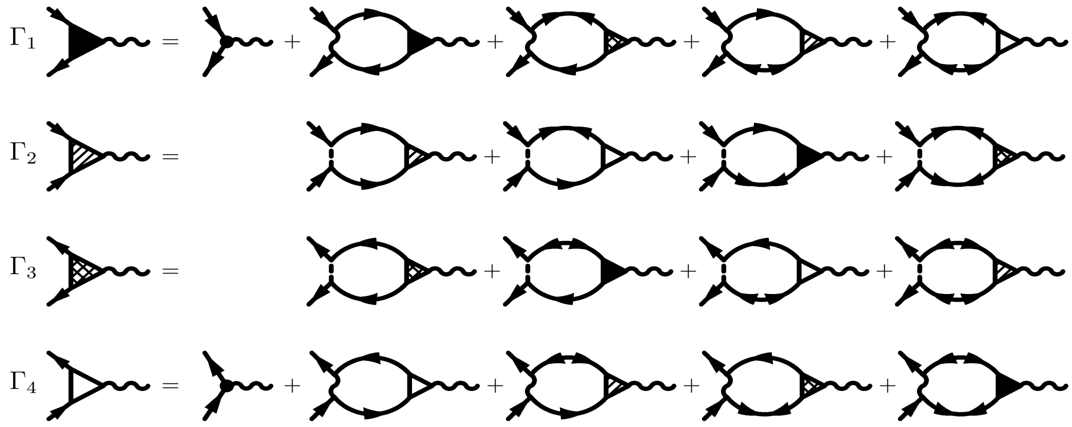

FIG. 1: A diagrammatic representation of the coupled integral equations for the effective weak vertices in superfluid baryonic matter. The "normal" propagators for particles (holes) are shown by single-arrowed lines directed from left to right (right to left). The double arrowed lines correspond to the "anomalous" propagators F (two incoming arrows) and F^+ (two outgoing arrows). The "normal" vertices Γ_1 and Γ_4 are shown by full and empty triangles. The "anomalous" vertices Γ_2 and Γ_3 are shown by hatched and shaded triangles. The horizontal wavy lines represent the low-energy propagator of Z^0 gauge boson. The vertical dashed lines stand for the particle-particle interaction v_{pp}; wavy lines represent particle-hole interaction v_{ph}.

integral equations [Ref. [5], Eqs. (20)-(22)]

$$\hat{\Gamma}_1 = \Gamma_0 + v_{ph}(G\Gamma_1 G + \hat{F}\hat{\Gamma}_3 G + G\hat{\Gamma}_2\hat{F} + \hat{F}\Gamma_4\hat{F}), \quad (9)$$

$$\hat{\Gamma}_2 = v_{pp}(G\hat{\Gamma}_2 G^+ + \hat{F}\Gamma_4 G^+ + G\Gamma_1\hat{F} + \hat{F}\hat{\Gamma}_3\hat{F}), \quad (10)$$

$$\hat{\Gamma}_3 = v_{pp}(G^+\hat{\Gamma}_3 G + \hat{F}\Gamma_1 G + G^+\Gamma_4\hat{F} + \hat{F}\hat{\Gamma}_2\hat{F}), \quad (11)$$

$$\hat{\Gamma}_4 = \Gamma_0 + v_{ph}(G^+\Gamma_4 G^+ + \hat{F}\Gamma_1\hat{F} + \hat{F}\hat{\Gamma}_2 G^+ + G^+\hat{\Gamma}_3\hat{F}), \quad (12)$$

which are written in the operator form and are shown diagrammatically in Fig. 1. The $\mu = 0$ index on vertices is suppressed, the bare vertex is $\Gamma_0 = 1$, and the hats over the vertices indicate that these are 2×2 matrices in the spin-space, specifically $\hat{\Gamma}^{(2)} = -i\sigma_y\Gamma^{(2)}$ and $\hat{\Gamma}^{(3)} = -i\sigma_y\Gamma^{(3)}$.

The vertices $\Gamma_1 \dots \Gamma_4$ are functions of the four-momentum transfer q and are independent of the incoming and outgoing momenta. Therefore, each term on the right hand side of Eqs. (9)-(12) contains a polarization insertion defined as [Ref. [5], Eq. (25)]

$$\Pi_{XX'}(q) = T\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\sum_{ip_n} X(p)X'(p+q) = \frac{\nu(0)}{4}\int_{-1}^{1} dx(X*X'), \quad (13)$$

where T is the temperature, $X, X' \in \{G, F, G^+, F^+\}$, $p \equiv (ip_n, \boldsymbol{p})$ with p_n being the fermionic Matsubara frequency, $\nu(0) = m^* p_F/\pi^2$ is the density of states at the Fermi surface, and x is the cosine of the angle formed by the vectors $\boldsymbol{q}$ and $\boldsymbol{p}$. The convolution (or loop) is defined in Eq. (13) as

$$(X*X') = T\int_{-\infty}^{\infty} d\xi_p \sum_{ip_n} X(p)X'(p+q). \quad (14)$$

Taking the integration limits over the infinite range $-\infty \le \xi_p \le \infty$ is a valid approximation in the weakly coupled superconductivity. One may now exploit relations among the loops [Ref. [5], Equations (26) and (A8)]: $(G*F) \simeq -(F*G)$, $(G^+*G) \simeq (G*G^+)$, and $(F*G^+) \simeq -(G^+*F)$ to show that Eqs. (10) and (11) have a solution only if $\Gamma_2 + \Gamma_3 = 0$, i. e., $\Gamma_2 = -\Gamma_3$. Eqs. (9) and (12) transform into each other on reversal of the time direction, which implies that $\Gamma_1(\omega, \boldsymbol{q}) = \mathcal{T}\Gamma_4(\omega, \boldsymbol{q})$, where $\mathcal{T}$ is an operator affecting the reversal. It is $+1$ for the scalar vertex and -1 for any vector vertex. This property is most easily checked by a calculation of the loops that change their sign, e. g. $(F*F)$, for scalar and vector vertices. The solution of the linear system of equations for the temporal part of the vector-current vertex (9)-(12) reads

$$\Gamma_1 = \frac{\Gamma_0\mathcal{C}}{\mathcal{C} - v_{ph}(\mathcal{A}^+\mathcal{C} - \mathcal{B}\mathcal{D}^+)}, \quad (15)$$

$$\Gamma_2 = -\frac{\Gamma_0\mathcal{D}^+}{\mathcal{C} - v_{ph}(\mathcal{A}^+\mathcal{C} - \mathcal{B}\mathcal{D}^+)}, \quad (16)$$

where $\mathcal{A}^{\mathcal{T}} = \Pi_{GG} - \Pi_{FF}\mathcal{T}$, $\mathcal{B} = 2\Pi_{FG}$, $\mathcal{C} = -[v_{pp}^{-1} - (\Pi_{GG^+} + \Pi_{FF})]$ and $\mathcal{D}^{\mathcal{T}} = \Pi_{FG^+}\mathcal{T} + \Pi_{GF}$. For $\mathcal{T} = 1$ one finds $\mathcal{B} = -\mathcal{D}^+$. Upon setting $\Gamma_0 = 1$ we verify that the solutions (15) and (16) coincide with those given in Ref. [5], Eqs. (33) and (34). The vector vertex $\vec{\Gamma}$ is defined in full analogy to the scalar vertex:

$$\vec{\Gamma}_1 = \vec{\Gamma}_0 + v_{ph}(G\vec{\Gamma}_1 G - F\vec{\Gamma}_3 G - G\vec{\Gamma}_2 F - F\vec{\Gamma}_4 F), \quad (17)$$

$$\vec{\Gamma}_2 = v_{pp}(G\vec{\Gamma}_2 G^+ + F\vec{\Gamma}_4 G^+ + G\vec{\Gamma}_1 F - F\vec{\Gamma}_3 F), \quad (18)$$

$$\vec{\Gamma}_3 = v_{pp}(G^+\vec{\Gamma}_3 G + F\vec{\Gamma}_1 G + G^+\vec{\Gamma}_4 F - F\vec{\Gamma}_2 F), \quad (19)$$

$$\vec{\Gamma}_4 = \vec{\Gamma}_0 + v_{ph}(G^+\vec{\Gamma}_4 G^+ - F\vec{\Gamma}_1 F - F\vec{\Gamma}_2 G^+ - G^+\vec{\Gamma}_3 F), \quad (20)$$

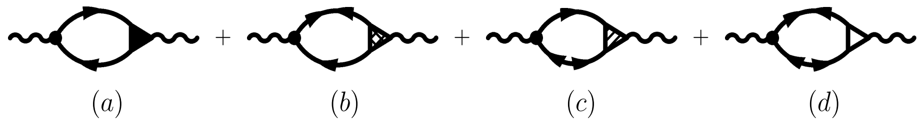

FIG. 2: The sum of polarization tensors contributing to the vector-current neutrino emission rate. Note that the diagrams b, c, and d are specific to the superfluid systems and vanish in the unpaired state.

where the same arguments as above imply $\vec{\Gamma}_2 = -\vec{\Gamma}_3$ and $\vec{\Gamma}_1 = \mathcal{T}\vec{\Gamma}_4$. There are two vectors available in the problem, therefore the most general decomposition of any vector vertex in terms of these vectors reads

$$\vec{\Gamma}_{1,2} = P_1\hat{p} + Q_{1,2}\hat{q}, \quad \vec{\Gamma}_0 = P_1\hat{p}, \tag{21}$$

where $\hat{p} = \boldsymbol{p}/|\boldsymbol{p}|$, $\hat{q} = \boldsymbol{q}/|\boldsymbol{q}|$ and P_1 and $Q_{1,2}$ are scalar functions, which depend on the moduli of these vectors. The solution of the set of linear equations (17)-(12) implies that

$$Q_1 = \frac{v_{ph}P_1[\tilde{\mathcal{A}}\mathcal{C} - \mathcal{B}\tilde{\mathcal{D}}]}{\mathcal{C} - v_{ph}[\mathcal{A}^+\mathcal{C} - \mathcal{B}\mathcal{D}^+]}, \tag{22}$$

$$Q_2 = -P_1\frac{\tilde{\mathcal{D}} - v_{ph}[\mathcal{A}^+\tilde{\mathcal{D}} - \tilde{\mathcal{A}}\mathcal{D}^+]}{\mathcal{C} - v_{ph}[\mathcal{A}^+\mathcal{C} - \mathcal{B}\mathcal{D}^+]}, \tag{23}$$

where $\tilde{\mathcal{A}} \equiv \mathcal{A}^-(\hat{q}\cdot\hat{p})$ and $\tilde{\mathcal{D}} \equiv \mathcal{D}^-(\hat{q}\cdot\hat{p})$ are the first moments of the $\mathcal{A}^-$ and $\mathcal{D}^-$ integrals with respect to the cosine of the angle formed by the vectors $\boldsymbol{q}$ and $\boldsymbol{p}$.

B. Polarization tensors

The 00 component of the vector polarization tensor is given by the sum of four terms

$$\Pi_{00} = \Gamma_0(G\Gamma_1 G + \hat{F}\hat{\Gamma}_3 G + G\hat{\Gamma}_2\hat{F} + \hat{F}\Gamma_4\hat{F}), \tag{24}$$

which are shown diagrammatically in Fig. 2. Exploiting the relations $\Gamma_1 = \mathcal{T}\Gamma_4$ and $\Gamma_2 = -\Gamma_3$ one finds $\Pi_{00} = \Gamma_0(\mathcal{A}^+\Gamma_1 + \mathcal{B}\Gamma_2)$. Then, substituting the vertices (15) and (16), we obtain

$$\Pi_{00} = \frac{\Gamma_0(\mathcal{A}^+\mathcal{C} - \mathcal{B}\mathcal{D}^+)}{\mathcal{C} - v_{ph}(\mathcal{A}^+\mathcal{C} - \mathcal{B}\mathcal{D}^+)}. \tag{25}$$

Because $\mathcal{B} = -\mathcal{D}^+$ for $\mathcal{T} = 1$ one recovers the polarization tensor given by Eq. (35) of Ref. [5]. For the spatial components of the vector current polarization tensor one finds

$$\Pi_{ij} = \hat{p}_i v_F^2\mathcal{A}^-\hat{p}_j + \hat{p}_i v_F^2\frac{v_{ph}[\tilde{\mathcal{A}}\mathcal{C} - \mathcal{B}\tilde{\mathcal{D}}]}{\mathcal{C} - v_{ph}[\mathcal{A}^+\mathcal{C} - \mathcal{B}\mathcal{D}^+]}\mathcal{A}^+\hat{q}_j - \hat{p}_i v_F^2\mathcal{B}\frac{\tilde{\mathcal{D}} - v_{ph}[\mathcal{A}^+\tilde{\mathcal{D}} + \tilde{\mathcal{A}}\mathcal{D}^+]}{\mathcal{C} - v_{ph}[\mathcal{A}^+\mathcal{C} - \mathcal{B}\mathcal{D}^+]}\hat{q}_j, \tag{26}$$

where we approximated the value of the P_1 by v_F.

C. Expanding the polarization tensor

We will use a double-expansion in small parameters which arise in the limit $|\boldsymbol{q}| \to 0$ with other parameters held fixed. Contrary to Ref. [5], we use the "symmetric" kinematics, where perturbed quasiparticle energies in the normal and superconducting state are written, respectively, as $\xi_\pm = \xi_p \pm (\boldsymbol{v}\cdot\boldsymbol{q})/2$ and $\epsilon_\pm = \sqrt{\xi_\pm^2 + \Delta^2}$. The loops $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are products of the convolution $(F * F)$ and prefactors which depend only on the four-momentum transfer [Ref. [12], Eqs. (15) and (17)]. These pre-factors are expanded in the small parameter $\delta = |q|v_F/\omega$. The expansion of the convolution $(F * F)$ around $\xi_p \simeq \xi_- \simeq \xi_+$ is carried out with respect to the second small paramater $\eta = v_F|\boldsymbol{q}|/\xi_p$; for details see Appendix A. Specifically, we write the Taylor expansion as

$$(F * F^+) = \sum_n (F * F^+)_n \delta^n x^n, \tag{27}$$

where $x \equiv \hat{q}\cdot\hat{p}$, and the factor η^n/δ^n is absorbed in the definition of the coefficients $(F * F^+)_n$ of the Taylor expansion (27). Substituting the expressions for the functions $\mathcal{A}^+$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}^+$ in Eq. (25) we obtain the scalar polarization tensor truncated at order $O(\eta^6)$ and $O(\delta^6)$:

$$\begin{aligned}\frac{\Pi_{00}(q)}{2\nu(0)} &= \overline{\left[-\frac{x\delta}{1 - x\delta} - 1\right](F * F^+)} \\ &+ \frac{\overline{(F * F^+)}}{\overline{(1 - x^2\delta^2)(F * F^+)}}\,\overline{(1 + x\delta)(F * F^+)} \\ &\simeq -\frac{4\delta^4}{45}\left(1 + \frac{25\delta^2}{21}\right)(F * F^+)_0 - \frac{4\delta^6}{45}(F * F^+)_2 \\ &+ O(\delta^8),\end{aligned} \tag{28}$$

where $\overline{(\dots)} \equiv (1/2)\int_{-1}^{1} dx(\dots)$. Similar expansion for the spatial part of the vector current polarization tensor

(26) gives

$$\frac{\Pi_{ii}(q)}{2v_F^2\nu(0)} = \overline{\left[-\frac{x\delta}{1-x\delta}\right]}(F*F^+) + \frac{\overline{(x+x^2\delta)(F*F^+)}}{\overline{(1-x^2\delta^2)(F*F^+)}}\,\overline{x^2\delta(F*F^+)} \simeq -\left(\frac{2\delta^2}{9}+\frac{22\delta^4}{135}+\frac{74\delta^6}{567}\right)(F*F^+)_0 - \left(\frac{14\delta^4}{135}+\frac{286\delta^6}{2835}\right)(F*F^+)_2 + O(\delta^8). \quad (29)$$

In this last expression we need to keep only the terms up to δ^4, because the vector vertices are proportional to the Fermi velocity, i. e., a small parameter.

IV. VECTOR CURRENT NEUTRINO EMISSIVITY

To obtain the neutrino pair bremsstrahlung emissivity we contract the baryonic polarization tensor with the trace over the leptonic currents, see Eq. (3). The result can be cast as

$$\epsilon = \frac{G^2 c_V^2 N_f}{48\pi^4}\int_0^\infty d\omega g(\omega)\omega J(\omega), \quad (30)$$

where $c_V = 1$ for neutrons and $c_V = 0.08$ for protons, $N_f = 3$ is the number of neutrino flavors in the Standard Model, and

$$J(\omega) = \int_0^\omega d\boldsymbol{q}\boldsymbol{q}^2(\boldsymbol{q}^2-\omega^2)\mathrm{Im}\left[\Pi_{00}(\omega,q)-\Pi_{ii}(\omega,q)\right] = -\frac{8\omega^5\nu(0)v_F^4}{405}\mathrm{Im}(F*F^+)_0[1+\gamma v_F^2], \quad (31)$$

where $\boldsymbol{q}$ is the momentum transfer and the coefficient γ is defined by Eq. (A17) of Appendix A. Substituting this result in Eq. (30) we find

$$\epsilon = \frac{16G^2c_V^2\nu(0)v_F^4}{1215\pi^3}I(z)T^7, \quad (32)$$

where $z = \Delta/T$ and

$$I(z) = z^7\int_1^\infty \frac{dy\, y^5}{\sqrt{y^2-1}}f(zy)^2\left[1+\left(\frac{7}{33}+\frac{41}{77}\gamma\right)v_F^2\right]. \quad (33)$$

To order v_F^4 this result agrees with those obtained by Refs. [4, 6]. The numerical result at order $O(v_F^6)$ and their comparison to order $O(v_F^4)$ result are shown in Fig. 3. One observes that the corrections to the emissivity are small, i. e., the series expansion is justified. The emissivity turns out to be enhanced by about 10% for $v_F = 0.4$ and by about 20% for $v_F = 0.6$ in the temperature range $0 \leq \tau \leq 0.92$, where $\tau = T/T_c$ with T_c being the critical temperature of pairing phase transition. In the close vicinity of T_c the emissivity is reduced.

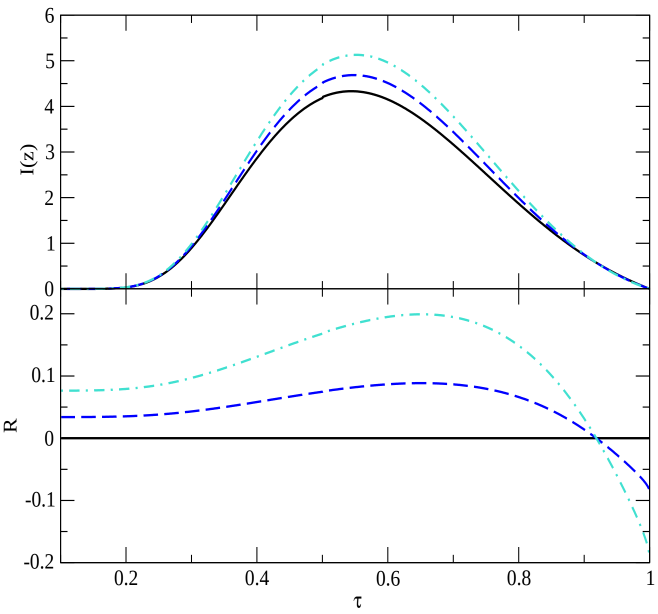

FIG. 3: (Color online) *Upper panel:* Dependence of the integral (33) on the reduced temperature τ for $v_F = 0$ (solid line), 0.4 (dashed line) and 0.6 (dashed dotted line). *Lower panel.* The ratio R of the integral (33) computed at $O(v_F^6)$ order to its value computed at order $O(v_F^4)$.

V. CONCLUDING REMARKS

We obtained the neutrino emissivity via neutral vector currents in a perturbative expansion up to order $O(v_F^6)$. The order $O(v_F^4)$ term corrects our previous NLO result [5] and is in agreement with the expressions given in Refs. [4, 6]. (The expansion of the vector polarization tensor in Ref. [5] gave a mathematically spurious NLO contribution that is linear in the nucleon recoil [19]. As a result, the emissivity was suppressed by a factor T/m, which arises from the recoil term, instead of the correct factor v_F^4).

From a practical point of applications in astrophysics, the $O(v_F^4)$ order term in the vector current neutrino emissivity is unimportant. Indeed, it is negligible compared to the axial vector neutrino emissivity as computed in Refs. [1, 3, 6], which remains unaffected by the vertex corrections. Nevertheless, the present reassessment removes the disagreement between Refs. [4, 6, 7] and Ref. [5], and confirms that the nonvanishing contribution to the vector-current emissivity arises at order $O(v_F^4)$. Similarly, it revises the results of Ref. [18] concerning the density response functions of cold neutron matter. However, we note that the absence of $O(v_F^2)$ terms in the vector response is not protected by any symmetry of the theory (i.e., conservation law), whereas the $O(1)$ contribution remaining in one-loop calculations are prohibited by baryon number conservation.

The computation of the $O(v_F^6)$ order contribution to the vector-current neutrino emissivity shows that the corrections to the leading non-zero term are below 10% for values $v_F \leq 0.4$ characteristic of baryons in compact stars. This result provides an evidence of the convergence of the series expansion of the vector-current polarization tensor in the regime where the momentum transfer is small compared to other relevant scales.

Acknowledgments

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Appendix A: Expansion of the loop function

The purpose of this appendix is to determine the coefficients of the expansion of the $(F*F)$ loop in powers of the η parameter, Eq. (27). The Matsubara sum for the FF loop gives (Eq. (A12) of Ref. [5])

$$
\begin{aligned}
T\sum_{ip} F(ip,\boldsymbol{p})F^{+}(ip+iq,\boldsymbol{k}) \;=\; & u_p u_k v_p v_k \Bigg\{ \left[\frac{1}{iq+\epsilon_p-\epsilon_k} - \frac{1}{iq-\epsilon_p+\epsilon_k}\right][f(\epsilon_p)-f(\epsilon_k)] \\
& + \left[\frac{1}{iq+\epsilon_p+\epsilon_k} - \frac{1}{iq-\epsilon_p-\epsilon_k}\right][1-f(\epsilon_p)-f(\epsilon_k)]\Bigg\}. \qquad \text{(A1)}
\end{aligned}
$$

For contact pairing interactions $u_p u_k v_p v_k = \Delta^2/4\epsilon_p\epsilon_k$. After an analytical continuation with retarded boundary condition, $i\omega_n \to \omega + i\delta$, we obtain for the convolution derived from Eq. (A1)

$$
(F*F^{+}) \;=\; -\frac{\Delta^2}{2}\int d\xi_p \Bigg\{ \frac{(\epsilon_p-\epsilon_k)}{\epsilon_p\epsilon_k}\left[\frac{f(\epsilon_p)-f(\epsilon_k)}{\omega^2-(\epsilon_p-\epsilon_k)^2+i\delta}\right] + \frac{(\epsilon_p+\epsilon_k)}{\epsilon_p\epsilon_k}\left[\frac{1-f(\epsilon_p)-f(\epsilon_k)}{\omega^2-(\epsilon_p+\epsilon_k)^2+i\delta}\right]\Bigg\}. \qquad \text{(A2)}
$$

To obtain the series expansion approximation to the integral on the right hand side we expand first the integrand in series and carry out the integrations term by term. In doing so we will need to compute the imaginary parts of integrals of the type

$$
I_n = \int_{-\infty}^{\infty} d\xi\; \frac{f_n(\epsilon)}{(\omega^2-4\epsilon^2)^n + i\delta}, \qquad \text{(A3)}
$$

where at the order we are working we need only integrals with $n = 1, 2$. For $n = 1$ the imaginary part of the integral is obtained with the use of the Dirac identity. Upon the following change of variables $\xi \to \epsilon = \sqrt{\xi^2+\Delta^2}$, $\xi d\xi = \epsilon d\epsilon$, $\Delta \leq \epsilon \leq \infty$, the integral acquires a factor of 2. A straightforward computation gives

$$
\begin{aligned}
\mathrm{Im} I_1 \;&=\; \mathrm{Im}\left[\int_{\infty}^{\infty} d\xi\; \frac{f_1(\epsilon)}{\omega^2-4\epsilon^2+i\delta}\right] \\
&=\; -\frac{\pi}{2}\frac{\theta(\omega-2\Delta)}{\sqrt{\omega^2-4\Delta^2}} f_1\left(\frac{\omega}{2}\right). \qquad \text{(A4)}
\end{aligned}
$$

Consider now the case $n = 2$, i. e., the integral

$$
\mathrm{Im} I_2 = \mathrm{Im}\left[\int_{\infty}^{\infty} d\xi\; \frac{f_2(\epsilon)}{(\omega^2-4\epsilon^2)^2+i\delta}\right]. \qquad \text{(A5)}
$$

We first change the variable $\xi \to \epsilon = \sqrt{\xi^2+\Delta^2}$, $\xi d\xi = \epsilon d\epsilon$ and rewrite the integral as

$$
\mathrm{Im} I_2 \;=\; 2\mathrm{Im}\left[\int_{\Delta}^{\infty} \frac{\epsilon d\epsilon}{\sqrt{\epsilon^2-\Delta^2}} \frac{f_2(\epsilon)}{(\omega^2-4\epsilon^2)^2+i\delta}\right] \qquad \text{(A6)}
$$

It is convenient to carry out a second transformation of variables defined as $z = 4\epsilon^2$, $dz = 8\epsilon d\epsilon$, $4\Delta^2 \leq z \leq \infty$. Implementing the transformation we obtain

$$
\mathrm{Im} I_2 \;=\; 2\pi\left[\int_{4\Delta^2}^{\infty} dz\; \frac{1}{4\sqrt{z-4\Delta^2}} f_2(\sqrt{z}/2)\delta^{(1)}(z-\omega^2)\right], \qquad \text{(A7)}
$$

where we used the formula

$$
\frac{1}{D^{k+1}+i\delta} = \frac{P}{D^{k+1}} - i\pi\frac{(-1)^k}{k!}\delta^{(k)}(D) \qquad \text{(A8)}
$$

where P denotes the principal value and $\delta^{(k)}(D)$ denotes the k-th derivative of the delta function. In the case $k = 1$ we obtain

$$
\frac{1}{D^2+i\delta} = \frac{P}{D^2} + i\pi\delta^{(1)}(D). \qquad \text{(A9)}
$$

We further use the formula

$$
\int f(x)\delta'(x-a) = -\int f'(x)\delta(x-a), \qquad \text{(A10)}
$$

to obtain the final result

$$
\begin{aligned}
\mathrm{Im} I_2 \;&=\; \mathrm{Im}\left[\int_{\infty}^{\infty} d\xi\; \frac{f_2(\epsilon)}{(\omega^2-4\epsilon^2)^2+i\delta}\right] \\
&=\; -\frac{\pi}{2}\theta(\omega-2\Delta)\frac{d}{dz}\left[\frac{f_2(\sqrt{z}/2)}{\sqrt{z-4\Delta^2}}\right]_{z=\omega^2}. \qquad \text{(A11)}
\end{aligned}
$$

To carry out the expansion we write the variables as

$$\xi_\pm = \xi(1 \pm \eta x), \quad \eta = \frac{qv_F}{\xi}, \tag{A12}$$

and substitute (A12) into the spectra $\epsilon_\pm = \sqrt{\xi_\pm^2 + \Delta^2}$. Expanding the kernel to leading order we obtain

$$(F * F^+)_0 \;=\; \Delta^2 \int_{-\infty}^{\infty} d\xi \frac{\tanh(\epsilon/2T)}{\epsilon(\omega^2 - 4\epsilon^2)}. \tag{A13}$$

Using Eq. (A4) one finds

$$\mathrm{Im}(F * F^+)_0 \;=\; -\frac{\pi\Delta^2}{\omega} \frac{\theta(\omega - 2\Delta)}{\sqrt{\omega^2 - 4\Delta^2}} \tanh\left(\frac{\omega}{4T}\right), \tag{A14}$$

where $\epsilon = \sqrt{\xi^2 + \Delta^2}$. The coefficient of the second order term is given by

$$\mathrm{Im}(F * F^+)_2 = -\frac{\pi\Delta^2}{\omega} \frac{\theta(\omega - 2\Delta)}{\sqrt{\omega^2 - 4\Delta^2}} [h_1(\omega) + h_2(\omega)],$$

where the contributions arising from integrals wit $n = 1$ and 2, respectively, are

$$h_1 \;\equiv\; \frac{1}{4y^2} \mathrm{sech}^2\left(\frac{\omega}{4\,T}\right) \left[\frac{\omega}{2T} + \left(2y^2 - 3\right) \sinh\left(\frac{\omega}{2T}\right) - \frac{\omega^2}{4T^2}\left(y^2 - 1\right) \tanh\left(\frac{\omega}{4T}\right)\right], \tag{A15}$$

$$h_2 \;=\; \frac{1}{4y^2} \mathrm{sech}^2\left(\frac{\omega}{4T}\right) \left[\frac{\omega}{2T} - \frac{\left(2y^2 - 3\right)}{y^2 - 1} \sinh\left(\frac{\omega}{2T}\right)\right]. \tag{A16}$$

Note that the function $h_{1,2}$ contain a factor $\eta^2/\delta^2 = (4y^2)/(y^2 - 1)$, with $y = \Delta/2\omega$. The ratio appearing in Eq. (33) is then given by

$$\gamma \;=\; \frac{\mathrm{Im}\,(F * F^+)_2}{\mathrm{Im}\,(F * F^+)_0} = \frac{1}{4y^2} \mathrm{sech}\left(\frac{\omega}{4T}\right) \mathrm{csch}\left(\frac{\omega}{4T}\right) \times \left[\frac{\omega}{T} + \frac{\left(y^2 - 2\right)}{y^2 - 1}\left(2y^2 - 3\right) \sinh\left(\frac{\omega}{2T}\right) - \frac{\omega^2}{4T^2}\left(y^2 - 1\right) \tanh\left(\frac{\omega}{4T}\right)\right]. \tag{A17}$$

Note that the second term in the brackets changes the singular asymptotics of the integrand in Eq. (33) at the end point $y = 1$ from $(y - 1)^{-1/2}$ to $(y - 1)^{-3/2}$.

Appendix B: Comparison with Ref. [5]

In this appendix the equations from Ref. [5] are preceded by roman I. As already mentioned above the polarization tensor (25) coincides with Eq. (I.35). We now proceed to examine the three loops appearing in these equations. Substituting Eqs. (I.A1) and (I.A3) into the expression for the $\mathcal{A}(q)$ loop and using the definitions of the Bogolyubov amplitudes appearing after Eq. (7) we obtain

$$\mathcal{A}(q) = \int \frac{d^3p}{(2\pi)^3} \left[(\epsilon + \epsilon')(\epsilon\epsilon' - \xi\xi' + \Delta^2) + \omega(\xi'\epsilon - \xi\epsilon')\right] \mathscr{G} \tag{B1}$$

with the short-hand notations $\xi = \xi_{\boldsymbol{p}}$, $\xi' = \xi_{\boldsymbol{p}+\boldsymbol{q}}$ (asymmetrical kinematics) or $\xi = \xi_{\boldsymbol{p}-\boldsymbol{q}/2}$, $\xi' = \xi_{\boldsymbol{p}+\boldsymbol{q}/2}$ (symmetrical kinematics), $\epsilon = \sqrt{\xi^2 + \Delta^2}$, $\epsilon' = \sqrt{(\xi')^2 + \Delta^2}$ and [20]

$$\mathscr{G} = \frac{1}{2\epsilon\epsilon'} \left[\frac{1 - f(\epsilon) - f(\epsilon')}{\omega^2 - (\epsilon + \epsilon')^2}\right]. \tag{B2}$$

On the other hand, substituting Eqs. (I.31) and (I.32) in Eq. (I.27), we see that the resulting expression does not contain the term $\omega(\xi'\epsilon - \xi\epsilon')$ in Eq. (B1), which vanishes in the limit $\boldsymbol{q} \to 0$. Using Eq. (I.A2) we further obtain

$$\mathcal{B}(q) = 2\Delta \int \frac{d^3p}{(2\pi)^3} \left[\omega\epsilon' + (\epsilon' + \epsilon)\xi'\right] \mathscr{G}. \tag{B3}$$

Substituting Eq. (I.31) in Eq. (I.28) we see that the term $(\epsilon' + \epsilon)\xi'$ in Eq. (B3) is missing. Finally, using (I.A3) and (I.A4) in the expression for the $\mathcal{C}$ loop we find

$$\mathcal{C}(q) \;=\; \int \frac{d^3p}{(2\pi)^3} \left\{\frac{1 - 2f(\epsilon)}{2\epsilon} - \left[(\epsilon + \epsilon')(\epsilon\epsilon' + \xi\xi' + \Delta^2) - \omega(\xi\epsilon' + \xi'\epsilon)\right] \mathscr{G}\right\} \tag{B4}$$

and we see that the term $-\omega(\xi\epsilon' + \xi'\epsilon)$ is missing in Eq. (I.29). Both terms that were dropped vanish in the limit $\boldsymbol{q} \to 0$, because they are odd in ξ while their convolutions involve integrations over symmetrical in ξ limits [see Eqs. (13) and Eqs. (14)]. Thus, we conclude that, while the $\boldsymbol{q} \to 0$ limit of the $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ loops were correctly identified, their small q expansion was carried out in Ref. [5] starting from incomplete expressions.

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[19] Linear in momentum transfer perturbation of quasiparticle energy is $(\boldsymbol{q}\cdot\boldsymbol{v}) \sim qv_F x$ [the recoil term is $O(q^2)$]. Any quantity which is expanded to the linear order in this perturbation will vanish upon integration over x in symmetrical limits. Therefore, truncating at linear order in the perturbation Ref. [5] do not find terms $\propto v_F$. In view of this observation the statement in Ref. [6] that the terms $\propto v_F$ were dropped in Ref. [5] should not be taken literally.

[20] The second term in Eq. (32) of Ref. [5] is a typographical error and should be dropped. It does not, however, affect the discussion or the result.